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Classification of 4-manifolds

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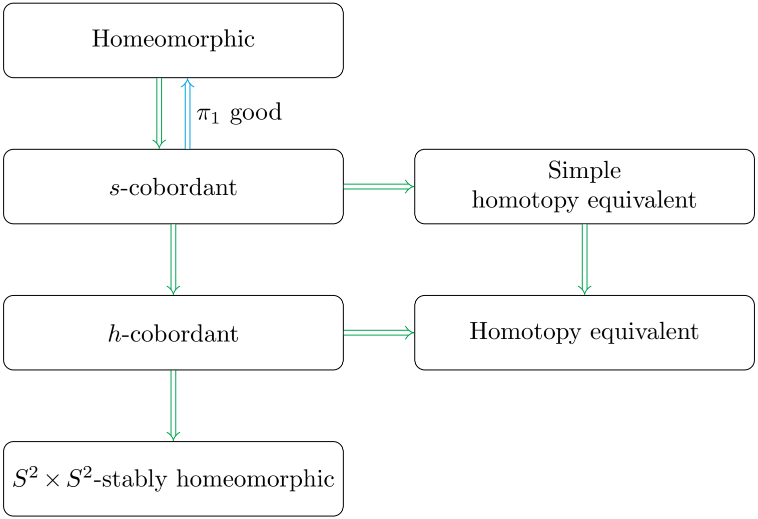
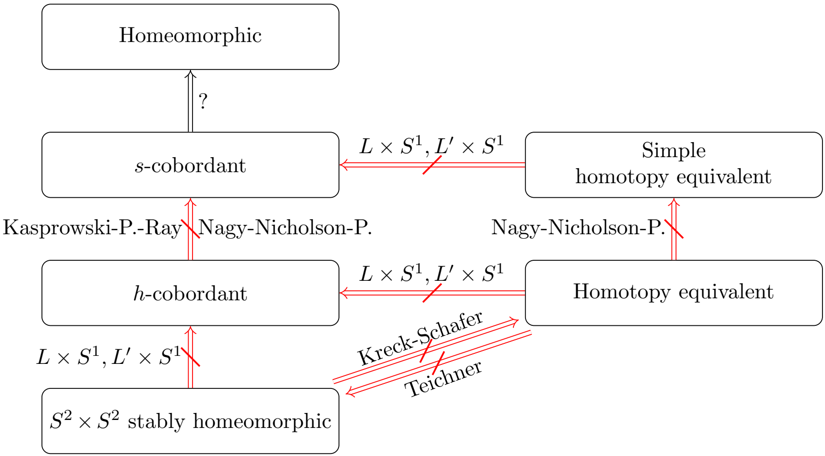
What classifications might be possible? What are the relationships between them?

In this talk X and X' will denote closed, oriented 4-manifolds. We do not assume manifolds are smooth unless stated.



Homeomorphic
π_1 good
s-cobordant
Simple homotopy equivalent
h-cobordant
Homotopy equivalent
$S^2 \times S^2$-stably homeomorphic



Homeomorphic
?
s-cobordant
$L \times S^1, L' \times S^1$
Simple homotopy equivalent
Kasprowski-P.-Ray
Nagy-Nicholson-P.
Nagy-Nicholson-P.
h-cobordant
$L \times S^1, L' \times S^1$
Homotopy equivalent
$L \times S^1, L' \times S^1$
Kreck-Schafer
Teichner
$S^2 \times S^2$ stably homeomorphic

Elaboration on two counterexamples.

Theorem (Atiyah-Bott, Turaev, Kwasik-Schultz)

Let L and L' be 3-dimensional lens spaces that are homotopy equivalent but not homeomorphic. Then $L \times S^1$ and $L' \times S^1$ are simple homotopy equivalent and stably homeomorphic but not h-cobordant.

Let E be the S^2-bundle over $\mathbb{RP}^2$ that is orientable but not spin.

We can define $*E$ via

$$E \# *\mathbb{CP}^2 \cong *E \# \mathbb{CP}^2.$$

Theorem (Teichner)

$E \# E$ and $*E \# *E$ are simple homotopy equivalent but not stably homeomorphic.

Theorem (Kasprowski-P.-Ray)

$*E \# *E$ is smoothable.

Now focus on classification up to homotopy equivalence and up to homeomorphism, starting with the simply-connected case.

Theorem (Whitehead 49)

Suppose X and X' are simply-connected. Then $\lambda_X \cong \lambda_{X'}$ if and only if $X \simeq X'$.

$$\begin{aligned} \lambda_X \colon H_2(X) \times H_2(X) &\to \mathbb{Z} \\ (x, y) &\mapsto \langle PD^{-1}(y), x \rangle \end{aligned}$$

Theorem (Freedman 82)

Suppose $X \simeq X'$ are simply-connected. Then $\mathrm{ks}(X) = \mathrm{ks}(X') \in \mathbb{Z}/2$ if and only if $X \cong X'$.

What is the analogue of $\lambda_X \colon H_2(X) \times H_2(X) \to \mathbb{Z}$ in the non-simply-connected case?

The *quadratic 2-type* is:

$$Q(X) = [\pi_1(X), \pi_2(X), k_X, \lambda_X].$$

Let $\pi := \pi_1(X)$. Then we consider $\pi_2(X)$ as a $\mathbb{Z}\pi$-module.

$$\lambda_X \colon \pi_2(X) \times \pi_2(X) \to \mathbb{Z}\pi$$

is the equivariant intersection form. Finally

$$k_X \in H^3(\pi; \pi_2(X))$$

classifies the fibration

$$K(\pi_2(X), 2) \to P_2(X) \to K(\pi, 1)$$

where $P_2(X)$ is the Postnikov 2-type.

If $Q(X) \cong Q(X')$ is $X \simeq X'$?

In general, no. For L and L' not homotopy equivalent lens spaces, but $\pi_1(L) \cong \pi_1(L')$, $Q(L \times S^1) \cong Q(L' \times S^1)$ but $L \times S^1$ and $L' \times S^1$ are not homotopy equivalent.

If $X \simeq X'$ and $\mathrm{ks}(X) = \mathrm{ks}(X')$, is $X \cong X'$?

In general no, as shown by $E \# E$ and $*E \# *E$.

But in many cases, yes; see the following table. But note extra invariant needed for $\pi = D_\infty$, as explained later. Also more is needed when $\partial X \neq \emptyset$.

Fundamental group	Closed	$\partial \neq \emptyset$	$\simeq$ or $\cong$	Realisation?
$\{1\}$	Freedman 82	Boyer 86	$\cong$	✓
$\mathbb{Z}$	Freedman-Quinn 90	Conway-P. 21 Conway-P.-Piccirillo 23 Conway-Kasprowski 25	$\cong$	✓
4-periodic cohomology	Hambleton-Kreck 88	?	$\simeq$	✗
$\mathbb{Z}/n$	Hambleton-Kreck 93	?	$\cong$	✓
Geom. 2-dim. WAA hyp.	Hambleton-Kreck-Teichner 08	Conway-Kasprowski 25	$\simeq + \cong$	✗
Solvable Baumslag-Solitar $\mathbb{Z} \ltimes \mathbb{Z}[\frac{1}{m}]$	Hambleton-Kreck-Teichner 08	Conway-Kasprowski 25	$\cong$	✓
2-Sylow $\mathbb{Z}/2^n \oplus \mathbb{Z}/2^m$	Kasprowski-P.-Ruppik 20	?	$\simeq$	✗
2-Sylow D_n	Kasprowski-Nicholson-Ruppik 21	?	$\simeq$	✗
π_1 aspherical 4-manifold	Kasprowski-Land 20 Degree one hyp.	Davis-Kasprowski-P. 25	$\simeq + \cong$	✗
$\pi_1(Y^3)$ s.t. finite subgroups cyclic	Hillman-Kasprowski-P.-Ray 25	Conway-Kasprowski 25	$\simeq$	✗
D_∞	Hillman-Kasprowski-P.-Ray 25	?	$\cong$	✗

TABLE 1. Known and soon to appear homotopy $\simeq$ and homeomorphism $\cong$ classifications of oriented 4-manifolds. The notation $\simeq + \cong$ indicates that the classification is a homotopy classification in general, and a homeomorphism classification for good groups. The realisation column asks whether it is known which values of $Q(X)$ are realised (plus any extra needed invariants when $\partial X \neq \emptyset$).

Next I will elaborate on the last two rows.

Theorem (Hillman-Kasprowski-P.-Ray)

Suppose $\pi = \pi_1(Y^3) = \pi_1(X) = \pi_1(X')$, *where* Y *is a closed, oriented 3-manifold, such that every finite subgroup of* π *is cyclic. Then*

$$Q(X) \cong Q(X') \Leftrightarrow X \simeq X'.$$

Theorem (Hillman-Kasprowski-P.-Ray)

If in addition π *is solvable and torsion-free, then there are at most two 4-manifolds, up to homeomorphism, with fixed* Q *and* ks.

Now restrict to $\pi = D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2 = \pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3)$, and seek a homeomorphism classification.

Need an extra invariant, $s(X) \in \mathbb{Z}/2 \times \mathbb{Z}/2$, due to Kreck-Lück-Teichner. Let $\widetilde{X}$ be the universal cover.

If $w_2(\widetilde{X}) \neq 0$, define $s(X) = 0$. If $w_2(\widetilde{X}) = 0$, consider

$$f \colon X \to BD_\infty \simeq \mathbb{RP}^\infty \vee [0,1] \vee \mathbb{RP}^\infty$$

and let $N := f^{-1}(1/2)$. Then $X = P_1 \cup_N P_2$. Since N lifts to $\widetilde{X}$ it is spin. Then N bounds a spin 4-manifold Q. Let

$$R_i := P_i \cup_N Q$$

for $i = 0, 1$. Define

$$s(X) = \big(\sigma(R_1)/8 + \mathrm{ks}(R_1), \sigma(R_2)/8 + \mathrm{ks}(R_2)\big) \in \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Let F be the unique spin S^2-bundle over $\mathbb{RP}^2$. The 4-manifolds E and $*E$ were defined above.

X	$E \# E$	$F \# F$	$*E \# *E$	$E \# *E$
$\mathrm{ks}(X)$	0	0	0	1
$s(X, \alpha)$	$(0, 0)$	$(0, 0)$	$(1, 1)$	$(0, 1)$

Theorem (Hillman-Kasprowski-P.-Ray)

Let X and X' be closed, oriented 4-manifolds with fundamental group D_∞. Suppose

1. $Q(X) \cong Q(X')$;
2. $\mathrm{ks}(X) = \mathrm{ks}(X') \in \mathbb{Z}/2$;
3. $s(X) = s(X') \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

Then $X \cong X'$.

Partial k-invariant free classification.

Theorem (Hillman-Kasprowski-P.-Ray)

Let X and X' be closed, oriented, smooth 4-manifolds with fundamental group D_∞. Suppose

$$\lambda_X \cong \lambda_{X'} \cong H(ID_\infty) \oplus \theta$$

where $H(ID_\infty)$ is the hyperbolic form on the augmentation ideal ID_∞ and θ is a form on a stably free $\mathbb{Z}D_\infty$-module. Then:

(i) $X \# \mathbb{CP}^2 \cong X' \# \mathbb{CP}^2$ (*or* $\overline{\mathbb{CP}}^2$).

(ii) *If the universal covers of X and X' are spin and the w_2-types in $H^2(D_\infty; \mathbb{Z}/2)$ are equal, then $X \cong X'$.*